Pergamon

Int. J. Solids Structures Vol. 35, No. 36, pp. 5189–5203, 1998

Printed in Great Britain
0020-7683/98/\$—see front matter

PII: S0020-7683(97)00263-1

A HIGHER ORDER SUBDOMAIN METHOD FOR FINDING LOCAL STRESS FIELDS IN COMPOSITES

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(*Received* 6 *March* 1997; *in revised form* 26 *August* 1997)

Abstract—The problem of finding local and volume averaged stresses in a two-dimensional heterogeneous solid is formulated in terms of fundamental point load solutions (Green's function) leading to singular integral equations. The resulting equations are solved approximately using a subdomain method in which closed form solutions for a rectangular subdomain are obtained and utilized to find the full field solution. Previously, closed form solutions for a rectangular subvolume had been found, but only for the case of an assumed constant strain. In the present paper the solution is obtained for a quadratic form which includes not only the usual constant term but also linear and quadratic terms. The advantages of using the higher order solutions is illustrated by finding the local field in a periodic composite with square fibers. The numerical solution takes less than 90 CPU s on a workstation. The solution yields average properties independent of the reference modulus as would be expected for an accurate solution of the singular integral equation and the effective transverse modulus vs volume fraction is close to that from Christensen's model developed for round fibers.

Keywords: Composites; Singular integral equation; Green's function; Eigenstrain; Contour integral; Subdomain.

1. INTRODUCTION

In the mechanics of heterogeneous materials there is an interest in computing the volume averaged properties from the geometry and constituent properties and in computing the local fields. Average stiffness properties are important in the design phase of a project and local fields are important for understanding local failure and damage. A wide variety of approaches have been used to obtain useful results, many of which are summarized in the following references: Aboudi (1991); Accorsi and Nemat-Nasser (1986); Bahei-El-Din *et al.* (1987); Christensen (1990); Mura (1987); Nemat-Nasser and Hori (1993); and Zhao and Weng (1990).

In the present work the problem is approached using fundamental solutions and singular integral equations (SIEs), i.e., via a Green's function approach. The challenge is in solving the singular integral equations for the geometries considered. For periodic composites representative volume elements (RVE) are used. For various specific cases previous researchers have obtained analytical solutions, Eshelby (1957), Mura (1987) or used numerical methods in which the strain is treated as a constant inside subvolumes that are square, Nemat-Nasser and Hori (1993), Walker *et al.* (1989), or triangular Walker *et al.* (1993). In the present work rectangular subdomains are used. However, the strain was represented as a quadratic polynomial. An analytical evaluation of the singular integrand function for a rectangular subvolume is then obtained for quadratic polynomial strains. This evaluation requires a special treatment of the singularity and the use of symbolic manipulation software. In evaluating the integral equation a small region around the singularity is treated using contour integration and that region is let shrink to zero. In the remaining region the result is obtained in closed form and when the singular region is shrunk to zero the integral in a Cauchy's principal value sense is obtained. The strain within each subdomain is expressed in terms of strains at selected collocation points so that a set

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of algebraic equations for the collocation point strains are readily obtained and solved. Thus, a polynomial representation of local strains is obtained in terms of collocation point strain values. Example results are computed for a layered composite by a quadratic polynomial approach and a periodic composite with rectangular fibers by three polynomial (quadratic, linear and constant) approaches. It is of interest to note that the effective transverse modulus and the volume average strain energy are independent of the reference material modulus choice, which is expected for accurate solutions. The results also compare favorably with the results of others.

2. THE SIE APPROACH

The local strain field, $\varepsilon_{ij}(\mathbf{r})$, at any point $\mathbf{r}(x_1, x_2, x_3)$ in the representative volume element (RVE) of an infinite, periodic fibrous lattice is determined, Walker *et al.* [1993, eqn (3.8)], by the singular integral equation:

$$\varepsilon_{ij}(\mathbf{r}) = \varepsilon_{ij}^0 - \iiint\limits_{\mathbf{V}} U_{ijkl}(\mathbf{r}-\mathbf{r}')\delta C_{klmn}(\mathbf{r}')\varepsilon_{mn}(\mathbf{r}')\,\mathrm{d}\mathbf{V}(\mathbf{r}')$$

$$+\frac{1}{\mathbf{V}_c}\iiint\limits_{\mathbf{V}_c}\mathrm{d}\mathbf{V}(\mathbf{r})\iiint\limits_{\mathbf{V}} U_{ijkl}(\mathbf{r}-\mathbf{r}')\delta C_{klmn}(\mathbf{r}')\varepsilon_{mn}(\mathbf{r}')\,\mathrm{d}\mathbf{V}(\mathbf{r}') \tag{1}$$

where ε_{ij}^0 are the strain tensor components applied to the infinite periodic lattice. $\mathbf{V}$ is the whole material domain, and $\mathbf{V}_c$ denotes the representative volume element (RVE). The RVE is the smallest representative unit in a periodic composite or for random composites a volume of sufficient size so as to yield averaged properties to the desired accuracy. $\mathbf{r}'(x_1', x_2', x_3')$ is the position vector representing the source point. The tensor components $\delta C_{klmn}(\mathbf{r}')$ are defined by the relation, $\delta C_{klmn}(\mathbf{r}') = C_{klmn}(\mathbf{r}') - C_{klmn}^0$, and give the component deviation of the elastic stiffness tensor at any field point $\mathbf{r}(x_1, x_2, x_3)$ from the homogeneous reference tensor components, C_{klmn}^0. The second term in eqn (1) represents the strain perturbation brought about by inserting the prismatic cylindrical fibers into an infinite homogeneous medium with reference elasticity components C_{klmn}^0, while the third term ensures that the volume average of the local strain fields, $\varepsilon_{ij}(\mathbf{r})$, over the RVE, is equal to the applied strain, ε_{ij}^0.

In the preceding equation,

$$U_{ijkl}(\mathbf{r}-\mathbf{r}') = -\frac{1}{2}\left(\frac{\partial^2 G_{ik}(\mathbf{r}-\mathbf{r}')}{\partial x_j \partial x_l} + \frac{\partial^2 G_{jk}(\mathbf{r}-\mathbf{r}')}{\partial x_i \partial x_l}\right) \tag{2}$$

are the fourth rank tensor components, which give the ij component of the strain at the field point $\mathbf{r}(x_1, x_2, x_3)$ due to the kl component of a stress applied at the source point $\mathbf{r}'(x_1', x_2', x_3')$ in an infinite homogeneous medium with the reference stiffness tensor components, C_{klmn}^0; $G_{ik}(\mathbf{r}-\mathbf{r}')$ are the Green's function components, shown in the Appendix for the isotropic material.

If the integration domain $\mathbf{V}$ contains the field point, $\mathbf{r}(x_1, x_2, x_3)$, i.e., $\mathbf{r}(x_1, x_2, x_3)$ coincides with source point $\mathbf{r}'(x_1', x_2', x_3')$, $U_{ijkl}(\mathbf{r}-\mathbf{r}')$ is singular. Thus, we have a governing singular integral equation.

3. EVALUATION OF THE SIE

Experimental results by Dow *et al.* (1966) and finite element calculations by Ghosh and Moorthy (1995) show that the local strain field over each phase, either matrix or fiber phase, changes smoothly. We therefore approximate the strain field over each subdomain by a quadratic polynomial, and assuming all values are independent of x_3', we have

$$\varepsilon_{ij}(x_1', x_2') = \eta_0 + \eta_1 x_2' + \eta_2 x_2'^2 + \eta_3 x_1' + \eta_4 x_1' x_2' + \eta_5 x_1' x_2'^2 + \eta_6 x_1'^2 + \eta_7 x_1'^2 x_2' + \eta_8 x_1'^2 x_2'^2, \quad (3)$$

where η_0–η_8 are nine constants.

The RVE is assumed to be divided into a total of p subdomains with V_k $(k = 1, 2, \ldots, p)$ denoting the volume of the kth subdomain. Each subdomain is further assumed to contain only one material phase. We now restrict our attention to the αth field point in subdomain k, $\mathbf{r}_\alpha(x_1^\alpha, x_2^\alpha, x_3^\alpha) \in \mathbf{V}_k$, and evaluate this singular integration in eqn (1) in two steps as follows. We first take a small prismatic cylinder $\mathbf{V}_{\text{square}}$: $|x_1^\alpha - x_1'| < \delta$, $|x_2^\alpha - x_2'| < \delta$, $-\infty < x_3' < \infty$, $\delta > 0$, surrounding the singular point, $\mathbf{r}_\alpha(x_1^\alpha, x_2^\alpha, x_3^\alpha)$ with its axis on the line $x_1' = x_1^\alpha$, $x_2' = x_2^\alpha$, $x_3' = \pm\infty$. If the prismatic cylinder is then shrunk to zero, i.e. $\delta \to 0$, the integral can be evaluated in Cauchy's principal value sense as follows:

$$\iiint_{\mathbf{V}_k} U_{ijkl}(\mathbf{r}_\alpha - \mathbf{r}')\delta C_{klmn}(\mathbf{r}')\varepsilon_{mn}(\mathbf{r}')\,\mathrm{d}\mathbf{V}(\mathbf{r}') = \lim_{\mathbf{V}_{\text{square}} \to 0}\left(\iiint_{\mathbf{V}_k - \mathbf{V}_{\text{square}}} U_{ijkl}(\mathbf{r}_\alpha - \mathbf{r}')\delta C_{klmn}(\mathbf{r}')\varepsilon_{mn}(\mathbf{r}')\,\mathrm{d}\mathbf{V}(\mathbf{r}') + \iiint_{\mathbf{V}_{\text{square}}} U_{ijkl}(\mathbf{r}_\alpha - \mathbf{r}')\delta C_{klmn}(\mathbf{r}')\varepsilon_{mn}(\mathbf{r}')\,\mathrm{d}\mathbf{V}(\mathbf{r}')\right). \quad (4)$$

(1) *Inside the prismatic cylinder*, $\boldsymbol{V}_{\text{square}}$

As the prismatic cylinder shrinks to zero, both the strain and the material properties over this small domain are constants due to continuity, and they can be taken as the values at that field point $\mathbf{r}_\alpha(x_1^\alpha, x_2^\alpha, x_3^\alpha)$.

Thus, we have the following:

$$\iiint_{\mathbf{V}_{\text{square}}} U_{ijkl}(\mathbf{r}_\alpha - \mathbf{r}')\delta C_{klmn}(\mathbf{r}')\varepsilon_{mn}(\mathbf{r}')\,\mathrm{d}\mathbf{V}(\mathbf{r}') = \iiint_{\mathbf{V}_{\text{square}}} U_{ijkl}(\mathbf{r}_\alpha - \mathbf{r}')\,\mathrm{d}\mathbf{V}(\mathbf{r}')\delta C_{klmn}(\mathbf{r}_\alpha)\varepsilon_{mn}(\mathbf{r}_\alpha). \quad (5)$$

The prismatic cylinder can contain a small enough circular cylinder $\mathbf{V}_\varepsilon$: $(x_1^\alpha - x_1')^2 + (x_2^\alpha - x_2')^2 = \varepsilon^2$, $-\infty < x_3' < \infty$, $0 < \varepsilon < \delta$, also with the line $x_1' = x_1^\alpha$, $x_2' = x_2^\alpha$, $x_3' = \pm\infty$, as its axis. Thus,

$$\iiint_{\mathbf{V}_{\text{square}}} U_{ijkl}(\mathbf{r}_\alpha - \mathbf{r}')\delta C_{klmn}(\mathbf{r}_\alpha)\,\mathrm{d}\mathbf{V}(\mathbf{r}') = \iiint_{\mathbf{V}_\varepsilon} U_{ijkl}(\mathbf{r}_\alpha - \mathbf{r}')\delta C_{klmn}(\mathbf{r}_\alpha)\,\mathrm{d}\mathbf{V}(\mathbf{r}') + \iiint_{\mathbf{V}_{\text{square}} - \mathbf{V}_\varepsilon} U_{ijkl}(\mathbf{r}_\alpha - \mathbf{r}')\delta C_{klmn}(\mathbf{r}_\alpha)\,\mathrm{d}\mathbf{V}(\mathbf{r}'). \quad (6)$$

Now the first term on the r.h.s. of eqn (6) can be found by using Eshelby's (1957) results as follows:

$$\iiint_{\mathbf{V}_\varepsilon} U_{ijkl}(\mathbf{r}_\alpha - \mathbf{r}')\delta C_{klmn}(\mathbf{r}_\alpha)\,\mathrm{d}\mathbf{V}(\mathbf{r}') = \iiint_{\mathbf{V}_\varepsilon} U_{ijrs}(\mathbf{r}_\alpha - \mathbf{r}')C^0_{rskl}C^{0\,-1}_{kluv}\delta C_{uvmn}(\mathbf{r}_\alpha)\,\mathrm{d}\mathbf{V}(\mathbf{r}')$$

$$= \iiint_{\mathbf{V}_\varepsilon} U_{ijrs}(\mathbf{r}_\alpha - \mathbf{r}')C^0_{rskl}\,\mathrm{d}\mathbf{V}(\mathbf{r}')C^{0\,-1}_{kluv}\delta C_{uvmn}(\mathbf{r}_\alpha) = S_{ijkl}C^{0\,-1}_{kluv}\delta C_{uvmn}(\mathbf{r}_\alpha), \quad (7)$$

where S_{ijkl} is the fourth-order Eshelby tensor.

Since the integrand function is continuous in the remaining domain, $\mathbf{V}_{\text{square}} - \mathbf{V}_\alpha$, and all values are independent of x_3', the second term on the r.h.s. of eqn (6) can be reduced to a two-dimensional integral and the contour integral then can be applied as follows:

$$\iiint\limits_{\mathbf{V}_{\text{square}}-\mathbf{V}_c} U_{ijkl}(\mathbf{r}_\alpha-\mathbf{r}')\delta C_{klmn}(\mathbf{r}_\alpha)\,\mathrm{d}\mathbf{V}(\mathbf{r}) = \iint\limits_{\mathbf{A}_{\text{square}}-\mathbf{A}_c} \mathrm{d}x'_1\,\mathrm{d}x'_2 \int_{x'_3=-\infty}^{x'_3=\infty} U_{ijkl}(\mathbf{r}_\alpha-\mathbf{r}')\delta C_{klmn}(\mathbf{r}_\alpha)\,\mathrm{d}x'_3$$

$$= \iint\limits_{\mathbf{A}_{\text{square}}-\mathbf{A}_c} \bar{U}_{ijkl}(x_1^\alpha-x'_1, x_2^\alpha-x'_2)\delta C_{klmn}(x_1^\alpha, x_2^\alpha)\,\mathrm{d}x'_1\,\mathrm{d}x'_2$$

$$= \oint\limits_{\mathbf{C}_{\text{square}}-\mathbf{C}_c} V_{ijkl}(x_1^\alpha-x'_1, x_2^\alpha-x'_2)\,\mathrm{d}x'_2\delta C_{klmn}(x_1^\alpha, x_2^\alpha), \quad (8)$$

where the fourth-order tensor components $\bar{U}_{ijkl}$ and V_{ijkl} are given in the Appendix. The above contour integral for either a square or a circle is constant and given by eqns (A9) and (A10) in the Appendix.

(2) *Outside the prismatic cylinder*, $V_k - V_{\text{square}}$

$$\iiint\limits_{\mathbf{V}_k-\mathbf{V}_{\text{square}}} U_{ijkl}(\mathbf{r}_\alpha-\mathbf{r}')\delta C_{klmn}(\mathbf{r}')\varepsilon_{mn}(\mathbf{r}')\,\mathrm{d}\mathbf{V}(\mathbf{r}')$$

$$= \iint\limits_{\mathbf{A}_k-\mathbf{A}_{\text{square}}} \mathrm{d}x'_1\,\mathrm{d}x'_2 \int_{x'_3=-\infty}^{x'_3=\infty} U_{ijkl}(\mathbf{r}_\alpha-\mathbf{r}')\delta C_{klmn}(\mathbf{r}')\varepsilon_{mn}(\mathbf{r}')\,\mathrm{d}x'_3$$

$$= \iint\limits_{\mathbf{A}_k-\mathbf{A}_{\text{square}}} \bar{U}_{ijkl}(x_1^\alpha-x'_1, x_2^\alpha-x'_2)\delta C_{klmn}(x'_1, x'_2)\varepsilon_{mn}(x'_1, x'_2)\,\mathrm{d}\mathbf{A}(x'_1, x'_2)$$

$$= \left(\int_{x_1^l}^{x_1^\alpha-\delta} + \int_{x_1^\alpha+\delta}^{x_1^u}\right)\left(\int_{x_2^l}^{x_2^\alpha-\delta} + \int_{x_2^\alpha+\delta}^{x_2^u}\right)\bar{U}_{ijkl}(x_1^\alpha-x'_1, x_2^\alpha-x'_2)\delta C_{klmn}(x'_1, x'_2)\varepsilon_{mn}(x'_1, x'_2)\,\mathrm{d}x'_1\,\mathrm{d}x'_2, \quad (9)$$

where the area $\mathbf{A}_k$ is the projection of the volume $\mathbf{V}_k$ in the x'_1–x'_2 plane; $\mathbf{A}_k = (x_1^l < x_1 < x_1^u, x_2^l < x_2 < x_2^u) \in \mathbf{A}_c$, where the area $\mathbf{A}_c$ is the projection of $\mathbf{V}_c$, $\mathbf{A}_c = \mathbf{A}_1 + \mathbf{A}_2 + \cdots \mathbf{A}_k + \cdots + \mathbf{A}_p$, shown in Fig. 3; and x_1^l, x_1^u; x_2^l, x_2^u are the limits of sub-domain $\mathbf{A}_k$ in x_1, x_2 directions, respectively.

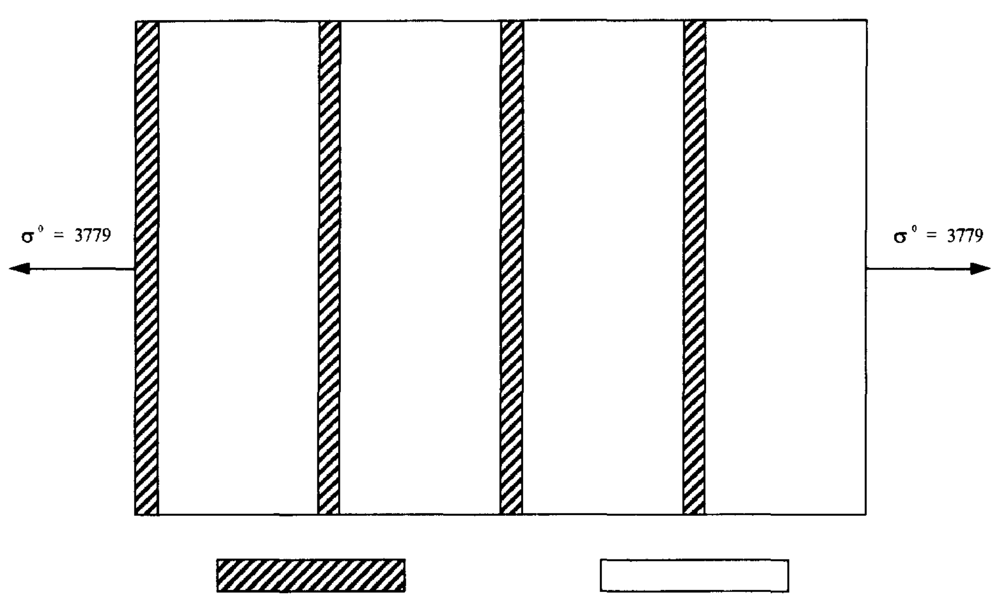

Fig. 1. A composite with layered structure [$\sigma_0(\times 10^4)$ Pa], $E_f/E_m = 3$, $\nu_f = \nu_m = 0.3$, $V_f = 0.1$.

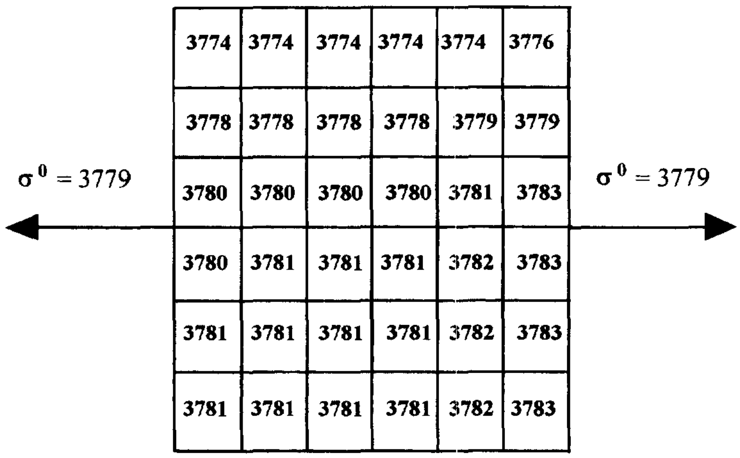

Fig. 2. Stress $\sigma_{11}(\times 10^4$ Pa) over a composite with layered structure.

Using a symbolic manipulation software, a closed form integration in eqn (9) can be obtained when the local strain $\varepsilon_{mn}(x_1', x_2')$ is represented as a quadratic polynomial in eqn (3). The limit value as $\delta \to 0$ in eqn (6) can then be obtained in closed form (see the Appendix).

For $\mathbf{r}(x_1^\alpha, x_2^\alpha, x_3^\alpha) \notin \mathbf{V}_k$ or $(x_1^\alpha, x_2^\alpha) \notin \mathbf{A}_k$, there are no singularities and the closed form evaluation in eqn (9) also applies.

Equations (7)–(9) can be added together to obtain the expression for the integral within a subdomain. For a RVE with multiple subdomains there will be, in addition to equations (7)–(9), contributions from subdomains 1 to p, and we obtain the integral expression covering the whole RVE,

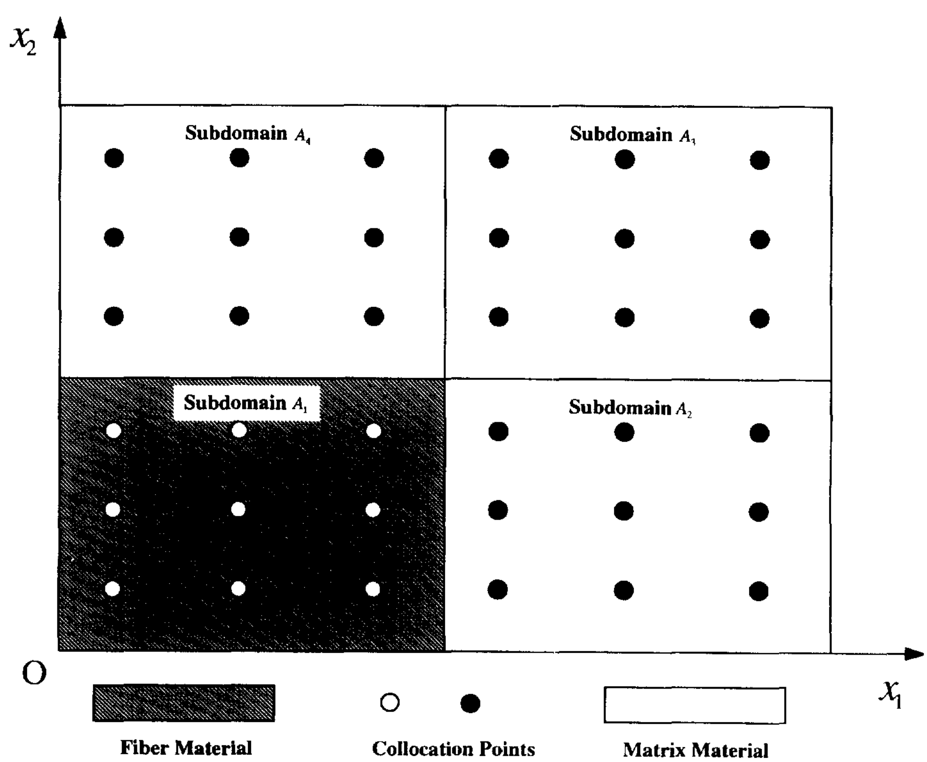

Fig. 3. The computation model of a quarter RVE.

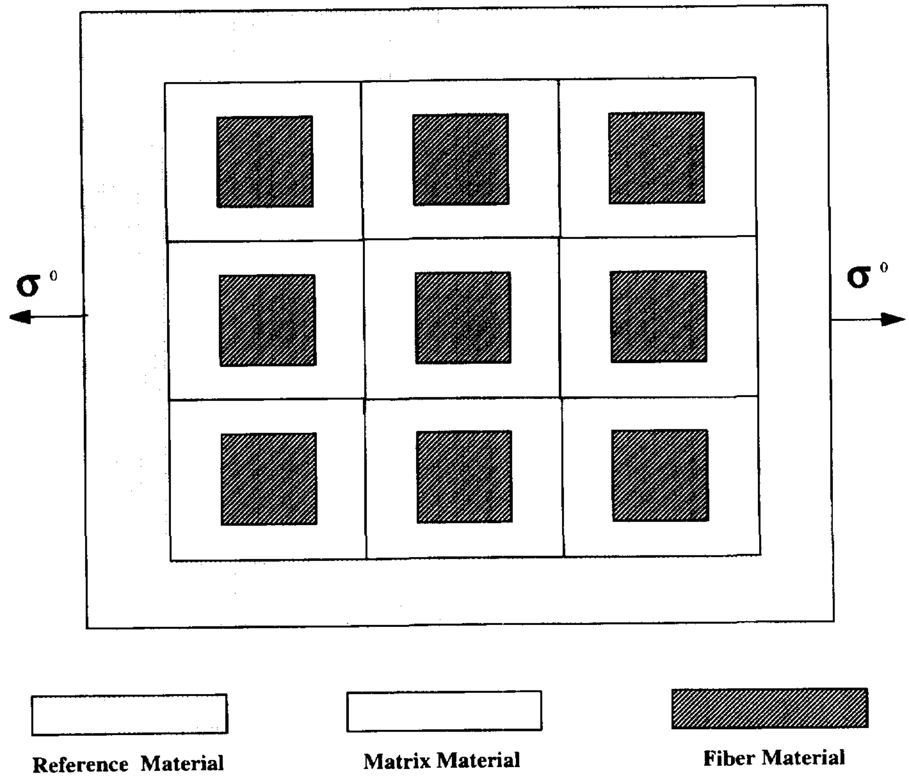

Fig. 4. The 2-D geometry of a periodic composite with square fibers.

$$\begin{aligned}
\iint_{\mathbf{A}_c} \bar{U}_{ijkl}(\mathbf{r}_\alpha-\mathbf{r}')\delta C_{klmn}(\mathbf{r}')\varepsilon_{mn}(\mathbf{r}')\,\mathrm{d}\mathbf{V}(\mathbf{r}') &= S_{ijkl}C^{0\,-1}_{kluv}\delta C_{uvmn}\varepsilon_{mn}(x_1^\alpha, x_2^\alpha) \\
&+ \oint_{\mathbf{C}_{\text{square}}-\mathbf{C}_\varepsilon} V_{ijkl}(x_1^\alpha - x_1', x_2^\alpha - x_2')\,\mathrm{d}x_2'\delta C_{klmn}(x_1^\alpha, x_2^\alpha)\varepsilon_{mn}(x_1^\alpha, x_2^\alpha) \\
&+ \lim_{\mathbf{A}_{\text{square}}\to\mathbf{0}}\left[\iint_{\mathbf{A}_c-\mathbf{A}_{\text{square}}} \bar{\mathrm{U}}_{ijkl}(\mathrm{x}_1^\alpha-\mathrm{x}_1', \mathrm{x}_2^\alpha-\mathrm{x}_2')\delta \mathrm{C}_{klmn}(\mathrm{x}_1', \mathrm{x}_2')\varepsilon_{mn}(\mathrm{x}_1', \mathrm{x}_2')\,\mathrm{d}\mathbf{A}(x_1', x_2')\right] \\
&+ \sum_{q=1, q\neq k}^{p} \iint_{\mathbf{A}_q} \bar{\mathrm{U}}_{ijkl}(x_1^\alpha - x_1', x_2^\alpha - x_2')\delta C_{klmn}(x_1', x_2')\varepsilon_{mn}(x_1', x_2')\,\mathrm{d}\mathbf{A}_q(x_1', x_2'). \qquad (10)
\end{aligned}$$

Equation (10) is now known in closed form and covers the RVE. To solve for the strain one must deal with integration over the infinite body via the two-dimensional version of eqn (1). Here, we develop the two-dimensional version of eqn (1) for a periodic structure only. The results for other cases could be developed in an obvious manner. In numerical computation for the periodic structures or other cases, practical considerations require that the integration be treated in an average sense beyond some distance from the point of interest. Based on convergence studies for the periodic structures studied here and elsewhere by Walker *et al.* (1989, 1990a, b, 1993), we find that explicit computations with the first nearest neighbors is adequate and generally is within 1% of calculations involving more distant neighbors. We note that for all subdomains outside of the RVE of interest the integration yields only the last term in eqn (10) for each subdomain because they contain no singularities. The second term in eqn (1) becomes the average of all the terms from eqn

(10) plus the summation terms from the neighboring RVEs with p subdomains, such that the second term of eqn (1) is as follows:

$$\iint_{\mathbf{A}} \bar{U}_{ijkl}(x_1^\alpha - x_1', x_2^\alpha - x_2')\delta C_{klmn}(x_1', x_2')\varepsilon_{mn}(x_1', x_2')\,\mathrm{d}\mathbf{A}(x_1', x_2')$$

$$= \iint_{\mathbf{A}_c} \bar{U}_{ijkl}(x_1^\alpha - x_1', x_2^\alpha - x_2')\delta C_{klmn}(x_1', x_2')\varepsilon_{mn}(x_1', x_2')\,\mathrm{d}\mathbf{A}_c(x_1', x_2')$$

$$+ \sum_{m_1=0}^{\pm\infty} \sum_{m_2=0}^{\pm\infty}{}' \iint_{\mathbf{A}_c} \bar{U}_{ijkl}(x_1^\alpha - x_1' - 2m_1 l_1, x_2^\alpha - x_2' - 2m_2 l_2)\delta C_{klmn}(x_1', x_2')\varepsilon_{mn}(x_1', x_2')\,\mathrm{d}\mathbf{A}_c(x_1', x_2'), \tag{11}$$

where the primed sum in the second term on the r.h.s. in eqn (11) denotes the summation in which the term with both $m_1 = 0$ and $m_2 = 0$ is deleted, where m_1 and m_2 are the numbers of the nearby periodic RVEs and l_1 and l_2 are the dimensions of a quarter RVE.

The final expression of the integral equation [eqn (1)] in two-dimensional form is

$$\varepsilon_{ij}(x_1^\alpha, x_2^\alpha) = \varepsilon_{ij}^0 - \iint_{A(x_1', x_2')} \bar{U}_{ijkl}(x_1^\alpha - x_1', x_2^\alpha - x_2')\delta C_{klmn}(x_1', x_2')\varepsilon_{mn}(x_1', x_2')\,\mathrm{d}A(x_1', x_2')$$

$$+ \frac{1}{A_c} \iint_{A_c(x_1^\alpha, x_2^\alpha)} \mathrm{d}A_c(x_1^\alpha, x_2^\alpha) \iint_{A(x_1', x_2')} \bar{U}_{ijkl}(x_1^\alpha - x_1', x_2^\alpha - x_2')\delta C_{klmn}(x_1', x_2')\varepsilon_{mn}(x_1', x_2')\,\mathrm{d}A(x_1', x_2'). \tag{12}$$

Equation (12) is evaluated by utilizing eqns (10) and (11) and the results in the Appendix.

4. LOCAL FIELD DETERMINATION

The local strain field is described in terms of unknown coefficients η_0–η_8. However, it is more convenient to solve for unknown strains at chosen collocation points because the collocation point strains have an easily understood physical meaning. The sets of unknown strains are equivalent and easily interchangeable with the coefficients η_0–η_8 via eqn (3). The nine collocation source points, β_1–β_9 were chosen to be evenly distributed over each subdomain for the quadratic polynomial approach as shown in Fig. (3). In introducing these unknowns the integrals are expressed in terms of weights W which can be found from the closed form integration given in eqn (12). The initial expression is

$$\iint_{\mathbf{A}_c} \bar{U}_{ijkl}(x_1^\alpha - x_1', x_2^\alpha - x_2')\delta C_{klmn}(x_1', x_2')\varepsilon_{mn}(x_1', x_2')\,\mathrm{d}\mathbf{A}_c(x_1', x_2')$$

$$= \sum_{\beta=\beta_1}^{\beta=\beta_9} W_{ijmn}^{\alpha\beta}\varepsilon_{mn}^{\beta} = W_{ijmn}^{\alpha\beta_1}\varepsilon_{mn}^{\beta_1} + W_{ijmn}^{\alpha\beta_2}\varepsilon_{mn}^{\beta_2} + \cdots + W_{ijmn}^{\alpha\beta_9}\varepsilon_{mn}^{\beta_9}, \tag{13}$$

where the weights $W_{ijmn}^{\alpha\beta_1}, W_{ijmn}^{\alpha\beta_2}, \ldots, W_{ijmn}^{\alpha\beta_9}$ are fourth-order tensor components corresponding to strain tensor $\varepsilon_{mn}^{\beta_1}, \ldots, \varepsilon_{mn}^{\beta_9}$.

Equation (13) must be equal to the l.h.s. of eqn (11) and so from the closed form integration of eqn (11) we can obtain the weights. To do this we equate eqn (13) with eqn (11) for each type of term in eqn (3) one at a time beginning with the constant term, yielding the following equations from which we can find the weights,

$$\iint_{\mathbf{A}_c} \bar{U}_{ijkl}(x_1^\alpha - x_1', x_2^\alpha - x_2')\delta C_{klmn}(x_1', x_2') \cdot 1 \, \mathrm{d}\mathbf{A}(x_1', x_2') = W_{ijmn}^{\alpha\beta_1} \cdot 1 + W_{ijmn}^{\alpha\beta_2} \cdot 1 + \cdots + W_{ijmn}^{\alpha\beta_9} \cdot 1,$$

$$\iint_{\mathbf{A}_c} \bar{U}_{ijkl}(x_1^\alpha - x_1', x_2^\alpha - x_2')\delta C_{klmn}(x_1', x_2') \cdot x_2' \, \mathrm{d}\mathbf{A}(x_1', x_2') = W_{ijmn}^{\alpha\beta_1} x_2'(\beta_1) + W_{ijmn}^{\alpha\beta_2} x_2'(\beta_2)$$

$$+ \cdots + W_{ijmn}^{\alpha\beta_9} x_2'(\beta_9),$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\iint_{\mathbf{A}_c} \bar{U}_{ijkl}(x_1^\alpha - x_1', x_2^\alpha - x_2')\delta C_{klmn}(x_1', x_2') \cdot x_1'^2 x_2'^2 \mathbf{A}(x_1', x_2') = W_{ijmn}^{\alpha\beta_1} x_2'^2(\beta_1) x_2'^2(\beta_1)$$

$$+ W_{ijmn}^{\alpha\beta_2} x_1'^2(\beta_2) x_2'^2(\beta_2) + \cdots + W_{ijmn}^{\alpha\beta_9} x_1'^2(\beta_9) x_2'^2(\beta_9). \quad (14)$$

In eqn (14) the l.h.s. integrals are known and the weights can be found by inverting the matrix. Equation (12) then becomes:

$$\varepsilon_{ij}^\alpha = \varepsilon_{ij}^0 - \sum_{\beta=1}^{M} W_{ijkl}^{\alpha\beta} \varepsilon_{kl}^\beta + \sum_{\gamma=1}^{M} \sum_{\beta=1}^{M} f^\gamma W_{ijkl}^{\gamma\beta} \varepsilon_{kl}^\beta, \quad (15)$$

where α, β and γ, ranging from 1 to M, are the αth, βth and γth collocation points and f^γ is the volume fraction factor of the γth subdomain. The number of the total collocation points in the RVE is equal to $9p$ for the quadratic polynomial.

Rearranging eqn (15), we obtain the following set of linear algebraic equations:

$$\sum_{\beta=1}^{M} B_{ijkl}^{\alpha\beta} \varepsilon_{kl}^\beta = \varepsilon_{ij}^0, \quad \text{for } \alpha = 1, 2, \ldots, M, \quad \text{where} \quad B_{ijkl}^{\alpha\beta} = \delta^{\alpha\beta} I_{ijkl} + W_{ijkl}^{\alpha\beta} - \sum_{\gamma=1}^{M} f^\gamma W_{ijkl}^{\gamma\beta}, \quad (16)$$

with

$$I_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

Equation (16) can be further written in matrix form to form a set of linear equations with $9k$ unknown collocation point strains which can be readily solved by standard methods. Each matrix element $\alpha\beta$ of the matrix B consists of a 6×6 submatrix, in the form:

$$[B^{\alpha\beta}]\{\varepsilon^\beta\} = \{\varepsilon^0\}, \quad (17)$$

$$[B^{\alpha\beta}] = \begin{bmatrix} B_{11}^{\alpha\beta} & B_{12}^{\alpha\beta} & B_{13}^{\alpha\beta} & B_{14}^{\alpha\beta} & B_{15}^{\alpha\beta} & B_{16}^{\alpha\beta} \\ B_{21}^{\alpha\beta} & B_{22}^{\alpha\beta} & B_{23}^{\alpha\beta} & B_{24}^{\alpha\beta} & B_{25}^{\alpha\beta} & B_{26}^{\alpha\beta} \\ B_{31}^{\alpha\beta} & B_{32}^{\alpha\beta} & B_{33}^{\alpha\beta} & B_{34}^{\alpha\beta} & B_{35}^{\alpha\beta} & B_{36}^{\alpha\beta} \\ B_{41}^{\alpha\beta} & B_{42}^{\alpha\beta} & B_{43}^{\alpha\beta} & B_{44}^{\alpha\beta} & B_{45}^{\alpha\beta} & B_{46}^{\alpha\beta} \\ B_{51}^{\alpha\beta} & B_{52}^{\alpha\beta} & B_{53}^{\alpha\beta} & B_{54}^{\alpha\beta} & B_{55}^{\alpha\beta} & B_{56}^{\alpha\beta} \\ B_{61}^{\alpha\beta} & B_{62}^{\alpha\beta} & B_{63}^{\alpha\beta} & B_{64}^{\alpha\beta} & B_{65}^{\alpha\beta} & B_{66}^{\alpha\beta} \end{bmatrix}. \quad (18)$$

By solving this group of linear algebraic equations, we then obtain the local strain field ε_{ij}^β.

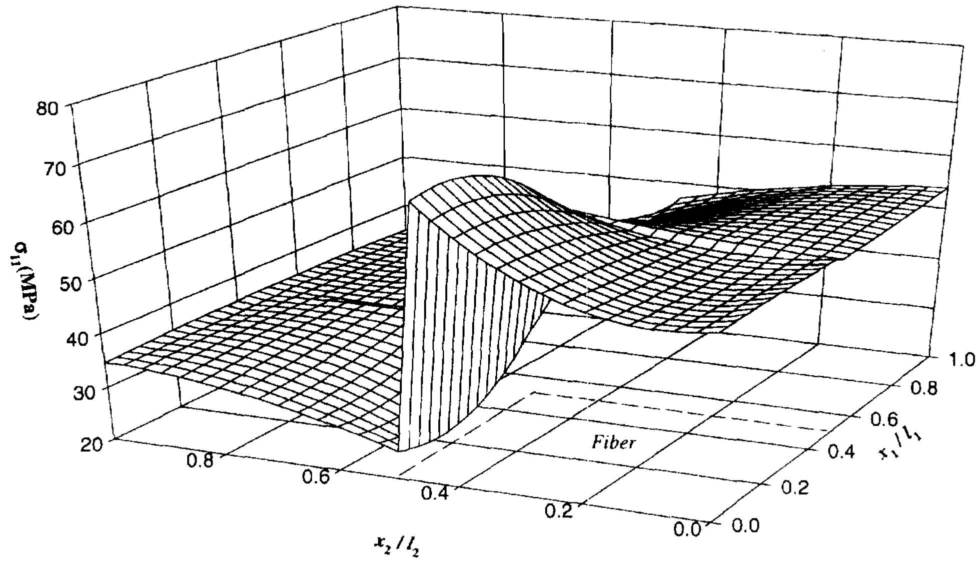

Fig. 5. Local stress σ_{11} by a quadratic polynomial approach, $E_f/E_m = 3$, $\nu_f = \nu_m = 0.3$, $V_f = 0.25$, fiber region: $0 < x_1 < 0.5$, $0 < x_2 < 0.5$.

5. EXAMPLE STUDIES

To test the accuracy of the method we apply this method to a layered composite structure for which the stresses should be exactly constant. Figure (1) shows the model, while the calculated stress field by a quadratic polynomial approach is shown in Fig. 2. Stresses are constant up to three digits showing good accuracy. From the calculation, the results are already convergent when first and second nearest neighbors were included.

We next apply this method to a periodic composite with rectangular fibers under uniaxial tension, Fig. 4, in which each phase is isotropic and homogeneous, with material properties given by $E_f/E_m = 3$, $\nu_f = \nu_m = 0.3$. Due to the geometry and loading symmetry, we need to solve only a quarter RVE.

A quarter RVE of the composite is represented by Fig. 3 where four subvolumes and 36 collocation points are used for the case of the quadratic polynomial. Figures 5–7 show the resultant local pull direction stress for solutions with the full quadratic polynomial, a linear polynomial, and subvolumes with only constant terms for a fiber volume fraction of 0.25 and the fiber region, $0 < x_1 < 0.5$, $0 < x_1 < 0.5$, as indicated by the dashed line. The case with constant terms had a number of subvolumes such that the total number of unknowns was the same as that in Fig. 5. It is readily apparent that the quadratic polynomial results in Fig. 5 are superior to the results from using the constant strain subvolumes.

The governing equation, eqn (12), if solved exactly, will yield results independent of the reference modulus. Figures 8 and 9 show that volume averaged quantities, strain energy and effective transverse modulus are virtually unaffected by the reference modulus chosen, and also vary by only 1% as the order of the polynomial changes.

Figure 10 shows the effective transverse modulus vs fiber volume fraction for the three different polynomial representations. It should be noted that there were 36 subvolumes for the constant polynomial, nine subvolumes for the linear polynomial and four subvolumes for the quadratic polynomial used in Fig. 10 such that the size of the matrix equation to be solved is the same for each case. For these polynomials the computed transverse moduli are within 1% of each other. A model with only four subvolumes and a single constant term representation of the strain was run and the effective transverse moduli vs fiber volume fraction were still within 2% of the moduli found using four subvolumes and quadratic polynomials. Thus, excellent volume averaged properties can be obtained from the simple models even at high volume fractions and in this case using only 4 CPU s. All cases are in reasonable agreement with Christensen's results for circular fibers. We note that the square

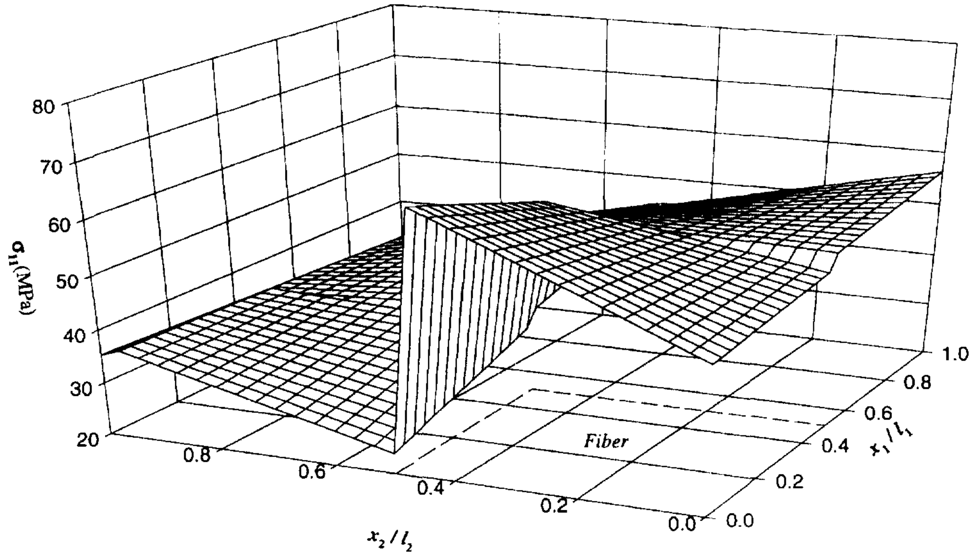

Fig. 6. Local stress σ_{11} by a linear polynomial approach, $E_f/E_m = 3$, $\nu_f = \nu_m = 0.3$, $V_f = 0.25$, fiber region: $0 < x_1 < 0.5$, $0 < x_2 < 0.5$.

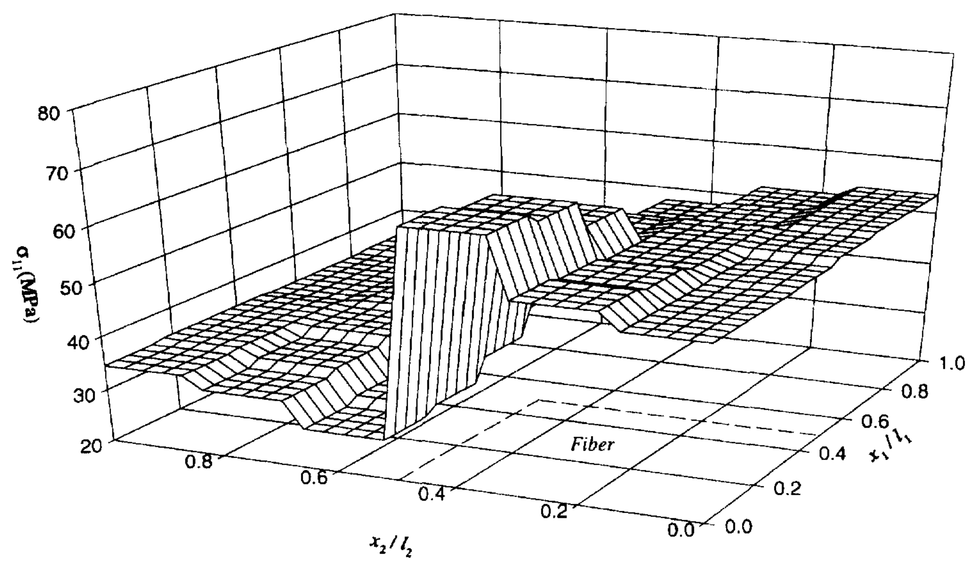

Fig. 7. Local stress σ_{11} by a constant polynomial approach, $E_f/E_m = 3$, $\nu_f = \nu_m = 0.3$, $V_f = 0.25$, fiber region: $0 < x_1 < 0.5$, $0 < x_2 < 0.5$.

fiber results presented in Fig. 10 are below the rule of mixtures, an upper bound, and above Christensen's results.

6. CONCLUSION

An efficient method was developed to determine the local elastic field and the overall elastic behavior of a heterogeneous medium based on the singular integral equation approach via a Green's function technique. Based on our previous work on subvolume techniques, the solution can be obtained by a contour integral and the Eshelby tensor. The singular integral can then be evaluated in closed form for assumed polynomial strain distributions with terms up to quadratic in the position coordinates on a rectangular

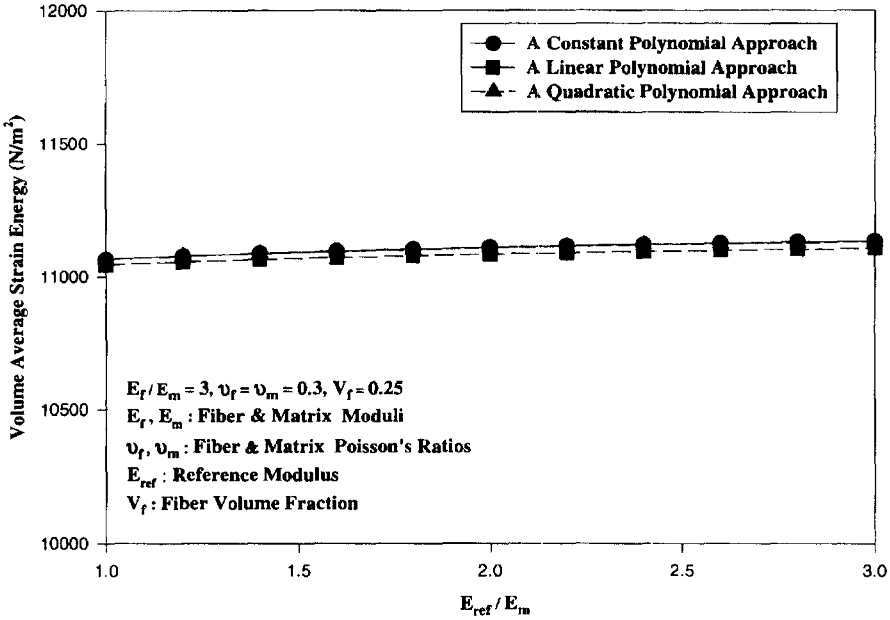

Fig. 8. Volume average strain energy vs reference modulus.

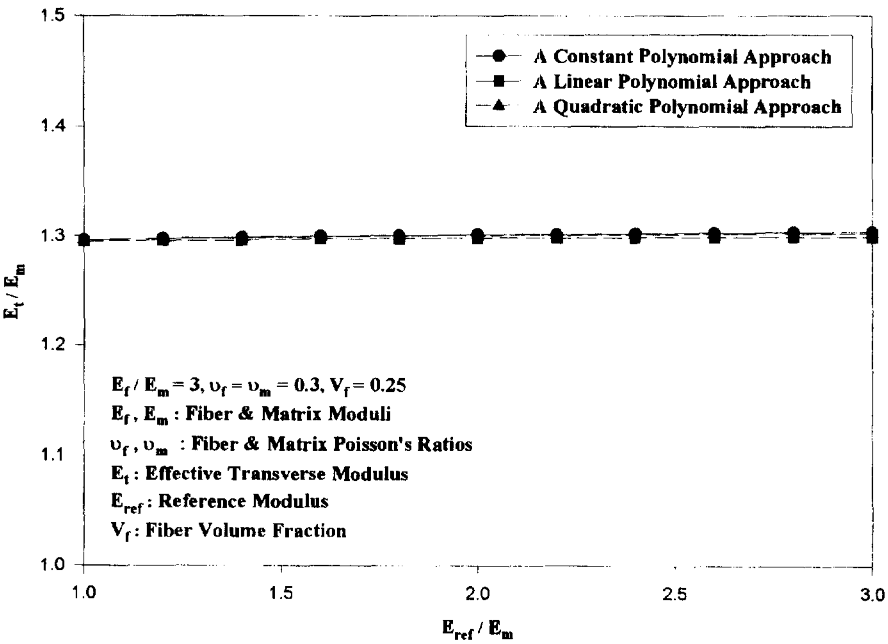

Fig. 9. Effective transverse modulus vs reference modulus.

subdomain. This closed form integration allowed the development of a higher order subvolume technique than the previous techniques which were based on assumed constant strain within the subvolume. The technique is used to solve a problem of a rectangularly packed composite with square fibers. It takes less than 90 CPU s on a Sun workstation for the numerical solution. The results are reasonable when compared to previous results. Volume average quantities vary by less than 1% with the choice of the reference modulus. The comparison of results using low order and high order polynomials shows that the simplest four-subvolume constant polynomial approximation yields excellent volume average quantities. The local stresses using different order polynomials are significantly different

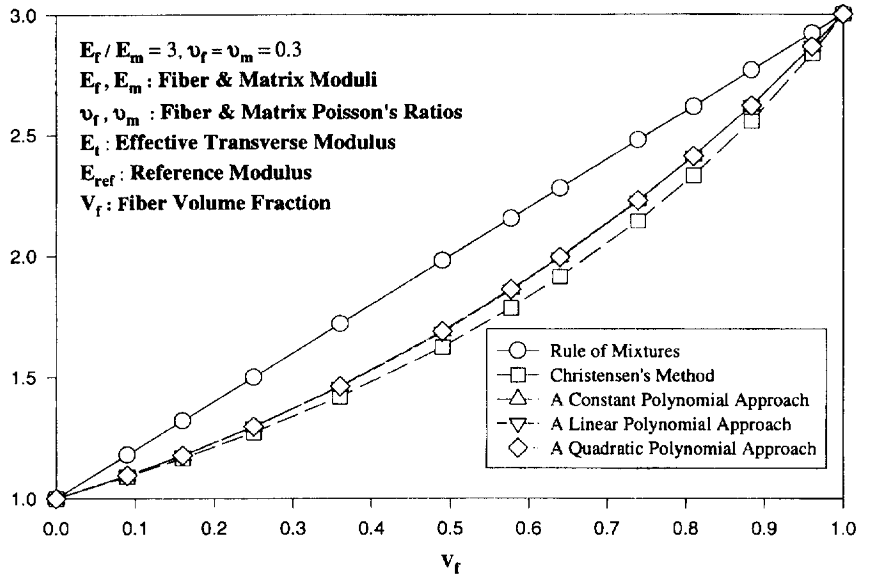

Fig. 10. Effective transverse modulus vs fiber volume fraction.

and improved with the higher order method when compared to the lower order methods using the same number degrees of freedom.

Acknowledgements—The work of the authors (J.C., E.H.J. and K.P.W.) was supported by the U.S. Department of Energy under Grant No. DE-FG02-92ER14247. Dr Oscar P. Manley served as the grant monitor.

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APPENDIX

1. $G_{jk}(\mathbf{r}_\alpha - \mathbf{r}')$

The Green's function components in eqn (2) for the isotropic reference material with constants λ^0 and μ^0 are as follows

$$G_{jk}(\mathbf{r}_\alpha - \mathbf{r}') = \frac{1}{8\pi\mu^0}\left(\delta_{jk}\frac{\partial^2|\mathbf{r}_\alpha - \mathbf{r}'|}{\partial x_q \partial x_q} - \frac{\lambda^0+\mu^0}{\lambda^0+2\mu^0}\frac{\partial^2|\mathbf{r}_\alpha - \mathbf{r}'|}{\partial x_j \partial x_k}\right) \tag{A1}$$

in which $|\mathbf{r}_\alpha - \mathbf{r}'| = \sqrt{X^2+Y^2+Z^2}$, where $X = x_1^\alpha - x_1'$, $Y = x_2^\alpha - x_2'$, $Z = x_3^\alpha - x_3'$.

2. $\bar{U}_{ijkl}(x_1^\alpha - x_1', x_2^\alpha - x_2')$

From eqns (4) and (A1), we have $\bar{U}_{ijkl}(x_1^\alpha - x_1', x_2^\alpha - x_2') = \int_{x_3'=-\infty}^{x_3'=\infty} U_{ijkl}(\mathbf{r}_\alpha - \mathbf{r}')\,\mathrm{d}x_3'$ and

$$U_{ijkl}(\mathbf{r}_\alpha - \mathbf{r}') = -\frac{1}{16\pi\mu^0}\left(\delta_{ik}\frac{\partial^4|\mathbf{r}_\alpha - \mathbf{r}'|}{\partial x_j \partial x_l \partial x_q \partial x_q} + \delta_{jl}\frac{\partial^4|\mathbf{r}_\alpha - \mathbf{r}'|}{\partial x_k \partial x_i \partial x_q \partial x_q} - \frac{2(\lambda^0+\mu^0)}{\lambda^0+2\mu^0}\frac{\partial^4|\mathbf{r}_\alpha - \mathbf{r}'|}{\partial x_i \partial x_j \partial x_k \partial x_l}\right). \tag{A2}$$

For example:

$$\bar{U}_{1111}(x_1^\alpha - x_1', x_2^\alpha - x_2') = -\frac{1}{16\pi\mu^0}(t_{1111}(x_1^\alpha - x_1', x_2^\alpha - x_2') + t_{1122}(x_1^\alpha - x_1', x_2^\alpha - x_2'))$$

$$+\frac{\lambda^0+\mu^0}{16\pi\mu^0(\lambda^0+2\mu^0)}t_{1111}(x_1^\alpha - x_1', x_2^\alpha - x_1'), \tag{A3}$$

where

$$t_{ijkl}(x_1^\alpha - x_1', x_2^\alpha - x_2') = \int_{x_3'=-\infty}^{x_3'=\infty}\frac{\partial^4|\mathbf{r}_\alpha - \mathbf{r}'|}{\partial x_i \partial x_j \partial x_k \partial x_l}\,\mathrm{d}x_3'. \tag{A4}$$

From eqns (A1)–(A3), it can be shown that

$$t_{1111}(X, Y) = \frac{2(X^4+6X^2Y^2-3Y^4)}{(X^2+Y^2)^3},$$

$$t_{2222}(X, Y) = \frac{-2(3X^4-6X^2Y^2-Y^4)}{(X^2+Y^2)^3},$$

$$t_{1122}(X, Y) = \frac{2(X^4-6X^2Y^2+Y^4)}{(X^2+Y^2)^3}. \tag{A5}$$

3. $V_{ijkl}(x_1^\alpha - x_1', x_2^\alpha - x_2')$

From eqn (11), we have the following relationship

$$\frac{\partial V_{ijkl}(x_1^\alpha - x_1', x_2^\alpha - x_2')}{\partial x_1'} = \bar{U}_{ijkl}(x_1^\alpha - x_1', x_2^\alpha - x'). \tag{A6}$$

Similar to the relationship between t_{ijkl} and $\bar{U}_{ijkl}$ in eqn (A3), the following is evident:

$$V_{1111}(x_1^\alpha - x_1', x_2^\alpha - x_2') = -\frac{1}{16\pi\mu^0}(q_{1111}(x_1^\alpha - x_1', x_2^\alpha - x_2') + q_{1122}(x_1^\alpha - x_1', x_2^\alpha - x_2'))$$

$$+\frac{\lambda^0+\mu^0}{16\pi\mu^0(\lambda^0+2\mu^0)}q_{1111}(x_1^\alpha - x_1', x_2^\alpha - x_2')), \tag{A7}$$

where

$$q_{1111}(x_1^\alpha - x_1', x_2^\alpha - x_2') = \int t_{1111}(s, x_2^\alpha - x_2')\,\mathrm{d}s = \frac{-2X(X^2+3Y^2)}{(X^2+Y^2)^2},$$

$$q_{2222}(x_1^\alpha - x_1', x_2^\alpha - x_2') = \int t_{2222}(s, x_2^\alpha - x_2')\,\mathrm{d}s = \frac{2X(3X^2+Y^2)}{(X^2+Y^2)^2},$$

$$q_{1122}(x_1^\alpha - x_1', x_2^\alpha - x_2') = \int t_{1122}(s, x_2^\alpha - x_2')\,\mathrm{d}s = \frac{2X(-X^2+Y^2)}{(X^2+Y^2)^2}, \tag{A8}$$

and:

$$\oint_{\mathbf{C}_{\text{square}}} q_{1111}(x_1^\alpha - x_1', x_2^\alpha - x_2')\,\mathrm{d}x_2' = 4(1-\pi),$$

$$\oint_{\mathbf{C}_{\text{square}}} q_{2222}(x_1^\alpha - x_1', x_2^\alpha - x_2')\,\mathrm{d}x_2' = 4(1-\pi),$$

$$\oint_{\mathbf{C}_{\text{square}}} q_{1122}(x_1^\alpha - x_1', x_2^\alpha - x_2')\,\mathrm{d}x_2' = -4, \tag{A9}$$

$$\oint_{\mathbf{C}_{\text{circle}}} q_{1111}(x_1^\alpha - x_1', x_2^\alpha - x_2')\,\mathrm{d}x_2' = -3\pi,$$

$$\oint_{\mathbf{C}_{\text{circle}}} q_{2222}(x_1^\alpha - x_1', x_2^\alpha - x_2')\,\mathrm{d}x_2' = -3\pi,$$

$$\oint_{\mathbf{C}_{\text{circle}}} q_{1122}(x_1^\alpha - x_1', x_2^\alpha - x_2')\,\mathrm{d}x_2' = -\pi. \tag{A10}$$

Combining eqns (A7)–(A10), we find that $\oint_{\mathbf{C}_{\text{square}}-\mathbf{C}_{\text{circle}}} V_{ijkl}(x_1^\alpha - x_1', x_2^\alpha - x_2')\,\mathrm{d}x_2'$ are constants.

4. $\iiint_{V_k - V_{\text{square}}} U_{ijkl}(\mathbf{r}_\alpha - \mathbf{r}')\delta C_{klmn}(\mathbf{r}')\varepsilon_{mn}(\mathbf{r}')\,\mathrm{d}\mathbf{V}(\mathbf{r}')$

As an example, here we show the procedure for the quadratic term. Two cases, i.e., the subdomain with and without the singularity, need to be considered.

(1) Subdomain with singularity: $(x_1^\alpha, x_2^\alpha) \notin \mathbf{A}_k$

From eqns (3) and (5), and the relationships between $\bar{U}_{ijkl}$ and t_{ijkl}, the integration with respect to $\bar{U}_{ijkl}x_1'^m \cdot x_2'^m$ can be reduced to that with respect to t_{ijkl} and X^mY^m, $m = 0, 1, 2$. The following, for example, gives the results for t_{1111} and X^2Y^2:

$$\int_c^d\int_a^b t_{1111}(X,Y)\cdot X^2Y^2\,\mathrm{d}X\,\mathrm{d}Y = \int_c^d\int_a^b \frac{2(X^4+6X^2Y^2-3Y^4)}{(X^2+Y^2)^3}\cdot X^2Y^2\,\mathrm{d}X\,\mathrm{d}Y$$

$$= 4a^4a\tan\left(\frac{c}{a}\right) - 4a^4a\tan\left(\frac{d}{a}\right) - 4b^4a\tan\left(\frac{c}{b}\right) + 4b^4a\tan\left(\frac{d}{b}\right)$$

$$-\frac{2ac^5}{a^2+c^2} + \frac{2ad^5}{a^2+d^2} - 4a^3c + 4a^3d + 2ac^3 - 2ad^3 + \frac{2bc^5}{b^2+c^2} + \frac{2bd^5}{b^2+d^2} - 4b^3c - 4b^3d - 2bc^3 + 2bd^3, \tag{A11}$$

where, $\mathbf{A}_k$: $x_1^l < x_1' < x_1^u$, $x_2^l < x_2' < x_2^u$, can be rewritten as $a < x_1' - x_1^\alpha < b$, $c < x_2' - x_2^\alpha < d$ and $a = x_1^l - x_1^\alpha$, $b = x_1^u - x_1^\alpha$; $c = x_2^l - x_2^\alpha$, $d = x_2^u - x_2^\alpha$.

(2) Subdomain with the singularity: $(x_1^\alpha, x_2^\alpha) \in \mathbf{A}_k$

The singularity part is discussed in the above and Section 2 and it is found to be a constant over a small square domain. Now we show how to evaluate the integration in the remaining subdomain $\mathbf{A}_k - \mathbf{A}_\varepsilon$ and obtain its limit value as $\mathbf{A}_\varepsilon \to 0$.

From eqn (12) we have the following:

$$\iiint_{\mathbf{V}_k - \mathbf{V}_{\text{square}}} U_{ijkl}(\mathbf{r}_\alpha - \mathbf{r}')\delta C_{klmn}(\mathbf{r}')\varepsilon_{mn}(\mathbf{r}')\,\mathrm{d}V$$

$$= \left(\int_{x_1^l}^{x_1^\alpha-\delta} + \int_{x_1^\alpha+\delta}^{x_1^u}\right)\left(\int_{x_2^l}^{x_2^\alpha-\delta} + \int_{x_2^\alpha+\delta}^{x_2^u}\right)\bar{U}_{ijkl}(x_1^\alpha - x_1', x_2^\alpha - x_2')\delta C_{klmn}(x_1', x_2')\varepsilon_{mn}(x_1', x_2')\,\mathrm{d}x_1'\,\mathrm{d}x_2'. \tag{A12}$$

Now the integration domain, $\mathbf{A}_k - \mathbf{A}_{\text{square}}$, is as follows:

$$(x_1' - x_1^\alpha, x_2' - x_2^\alpha) \in \{\langle(a, b), (c, d)\rangle - \langle(-\delta, \delta), (-\delta, \delta)\rangle\}. \tag{A13}$$

Similarly, we again use t_{1111} and X^2Y^2 as the example and the integration is reduced to the following:

$$\left(\int_c^d \int_a^b - \int_{-\delta}^{\delta}\int_{-\delta}^{\delta}\right)\frac{2(X^4+6X^2Y^2-3Y^4)}{(X^2+Y^2)^3}\cdot X^2Y^2\,\mathrm{d}X\,\mathrm{d}Y = \left(\int_c^{-\delta} + \int_\delta^d\right)\left(\int_a^{-\delta} + \int_\delta^b\right)\frac{2(X^4+6X^2Y^2-3Y^4)}{(X^2+Y^2)^3}\cdot X^2Y^2\,\mathrm{d}X\,\mathrm{d}Y$$

$$= \left(\int_c^{-\delta}\int_a^{-\delta} + \int_c^{-\delta}\int_\delta^b + \int_\delta^d\int_a^{-\delta} + \int_\delta^d\int_\delta^b\right)\frac{2(X^4+6X^2Y^2-3Y^4)}{(X^2+Y^2)^3}\cdot X^2Y^2\,\mathrm{d}X\,\mathrm{d}Y. \tag{A14}$$

Letting $\mathbf{A}_\varepsilon \to 0$, i.e. $\delta \to 0$, we have the following:

$$\lim_{\delta\to 0}\int_c^{-\delta}\int_a^{-\delta}\frac{2(X^4+6X^2Y^2-3Y^4)}{(X^2+Y^2)^3}\cdot X^2Y^2\,\mathrm{d}X\,\mathrm{d}Y = 4a^4 a\tan\left(\frac{c}{a}\right) - \frac{2a^3c(2a^2+c^2)}{a^2+c^2},$$

$$lim_{\delta\to 0}\int_\delta^d\int_a^{-\delta}\frac{2(X^4+6X^2Y^2-3Y^4)}{(X^2+Y^2)^3}\cdot X^2Y^2\,\mathrm{d}X\,\mathrm{d}Y = \frac{2a^3d(sa^2+d^2)}{a^2+d^2} - 4a^4 a\tan\left(\frac{d}{a}\right),$$

$$lim_{\delta\to 0}\int_c^{-\delta}\int_\delta^b\frac{2(X^4+6X^2Y^2-3Y^4)}{(X^2+Y^2)^3}\cdot X^2Y^2\,\mathrm{d}X\,\mathrm{d}Y = -4b^4 a\tan\left(\frac{c}{b}\right) + \frac{2bd^5}{b^2+c^2} + 4b^3c - 2bc^3,$$

$$lim_{\delta\to 0}\int_\delta^d\int_\delta^d\frac{2(X^4+6X^2Y^2-3Y^4)}{(X^2+Y^2)^3}\cdot X^2Y^2\,\mathrm{d}X\,\mathrm{d}Y = 4b^4 a\tan\left(\frac{d}{b}\right) - \frac{2bd^5}{b^2+d^2} - 4b^3d - 2bd^3. \tag{A15}$$

For the other terms, the same technique applies.